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Lecture 2

The fundamental curve of p-adic Hodge theory

Defined and studied in my joint work with Fontaine.

$\longrightarrow$ Will replace the proper smooth curve X from my preceding lecture to geometrize class field theory for a local field E.

$$E \nearrow \mathbb{F}_q((\pi))$$
$$E \searrow [E:\mathbb{Q}_p] < +\infty \qquad \mathcal{O}_E/\pi = \mathbb{F}_q$$

$F|\mathbb{F}_q$ perfectoid field

$\hookrightarrow$ Complete/non trivial $|\cdot| : F \to \mathbb{R}_+$
perfect

$\underline{\underline{\text{Ex}}}$: $\mathbb{F}_q((T^{1/p^\infty}))$, $\widehat{\mathbb{F}_q((T))}$

Define holomorphic functions of the variable π wrt. coefficients in F.

seen as an adic space

$$* \; E = \mathbb{F}_q((\pi)), \quad Y_F := \mathbb{D}^*_F = \{0 < |\pi| < 1\} \subset \mathbb{A}^1_F$$

punctured open disk $/F$

$$\mathcal{O}(Y_F) = \left\{ \sum_{n \in \mathbb{Z}} x_n \pi^n \;\middle|\; x_n \in F, \ \forall \rho \in]0,1[\ \lim_{|n| \to +\infty} |x_n| \rho^n = 0 \right\}$$

$$\begin{tikzcd}
 & \mathbb{D}^*_F \arrow[dl] \arrow[dr] & \\
\mathrm{Spa}(F) & & \mathbb{D}^*_{\mathbb{F}_q} = \mathrm{Spa}(E)
\end{tikzcd}$$

standard structural morphism

not locally top. of finite type structural morphism we are interested in

$\mathbb{F}_q((\pi)) \subset \mathcal{O}(Y_F)$

$$* \; E | \mathbb{Q}_p \qquad \underbrace{\mathbb{A}}_{\text{Fontaine's ring } A_{\inf}} = W_{\mathcal{O}_E}(\mathcal{O}_F)$$

$$= \left\{ \sum_{n \geq 0} [x_n] \pi^n \;\middle|\; x_n \in \mathcal{O}_F \right\}$$

$\varpi \in F,\ 0 < |\varpi| < 1$

$$\underbrace{\mathbb{A}\left[\frac{1}{\pi}, \frac{1}{[\varpi]}\right]} = \left\{ \sum_{n \gg -\infty} [x_n]\pi^n \,\middle|\, x_n \in F,\ \sup_n |x_n| < +\infty \right\}$$

holomorphic functions

meromorphic along (π) and $([\varpi])$.

$$\rho \in]0,1[\qquad \left|\sum_n [x_n]\pi^n\right|_\rho = \sup_n |x_n|\rho^n$$

$|\cdot|_\rho =$ Gauss norm radius ρ.

$\mathcal{O}(Y_F) :=$ Frechet E-algebra completion of $\mathbb{A}\left[\frac{1}{\pi}, \frac{1}{[\varpi]}\right]$
w.r.t. $(|\cdot|_\rho)_{\rho \in]0,1[}$

invert π and $[\varpi]$

$$\underbrace{Y_F} = \operatorname{Spa}(\mathbb{A}, \mathbb{A}) \setminus V(\pi[\varpi])$$

E-adic space "Stein"
Completely determined by $\mathcal{O}(Y_F)$

equipped with $(\pi, [\varpi])$-adic topology

the fact this is sheafy is a theorem

Rem: $E = \mathbb{F}_q((\pi))$ $\qquad Y_F = \operatorname{Spa}(F) \times \operatorname{Spa}(E)$ categorical product

When $E|\mathbb{Q}_p$ can give a meaning to this via Scholze's theory of diamonds

$$\rightsquigarrow \quad Y_F^\diamond = \operatorname{Spa}(F) \times_{\operatorname{Spa}(\mathbb{F}_q)} \operatorname{Spa}(E)^\diamond$$

The adic curve

$\varphi = \mathrm{Frob}_F$ induces an automorphism of Y_F

via $\varphi\left(\sum_n [x_n]\pi^n\right) = \sum_n [x_n^q]\pi^n$

$Y_F \circlearrowleft \varphi$ properly discontinuous without fixed points

$\rho \in]0,1[$, $|\varphi(f)|_\rho = |f|_{\rho^{1/q}}^q$

$\Downarrow$

$\varphi(\text{annulus}\{|\pi| = \rho\}) = \text{annulus}\{|\pi| = \rho^{1/q}\}$

Def: $X_F = Y_F / \varphi^{\mathbb{Z}}$ quasi-compact E-adic space $\underbrace{}_{\text{not of finite type}}$

Classical points:

Def: $\xi = \sum_{n\geq 0} [x_n]\pi^n \in A$ is primitive of degree $d \geq 1$ if $x_0 \neq 0$, $x_0, \dots, x_{d-1} \in \mathfrak{m}_F$ and $x_d \in \mathcal{O}_F^\times$

$\rightarrow$ notion from Weierstraß factorization theory.

degree d $\times$ degree d' = degree $d+d'$

Th: ξ irreducible primitive degree d.

(1) $K = \mathcal{O}(Y_F)/\xi$ is a perfectoid field $|E$ with

$F \hookrightarrow K^\flat$ satisfying $[K^\flat : F] = d$

$x \mapsto \left([x^{p^{-n}}] \bmod \xi\right)_{n\geq 0}$

(2) F alg. closed $\Rightarrow$ $d = 1$

$\rightsquigarrow$ Weierstraß factorization $\forall \xi$ primitive

$$\xi = \underset{A^\times}{u} \times (\pi - [a_1]) \times \dots \times (\pi - [a_d])$$

not unique if $E|\mathbb{Q}_p$.

(3) $\forall I = [\rho_1, \rho_2] \subset]0,1[$ compact

$$Y_I = \text{Compact annulus } \{|\pi| \in I\}$$

$\mathcal{O}(Y_I)$ is a P.I.D. with $\mathrm{Spm}(\mathcal{O}(Y_I))$

$$\|\xi\| = |x_0|^{1/\deg \xi} \qquad \qquad \| \qquad \{\xi \text{ primitive irred. s.t. } \|\xi\| \in I\}/A^\times$$

+ good theory of Newton polygons for elements of $\mathcal{O}(Y_F)$

Def: $|Y_F|^{cl} = \{V(\xi) \mid \xi \text{ primitive irred.}\} \subset |Y_F|$

"Classical Tate points"

$$|X_F|^{cl} = |Y_F|^{cl}/\varphi^{\mathbb{Z}}.$$

$$\boxed{\text{The schematical curve}}$$

$\mathcal{O}(1)$ line bundle $/X_F$

trivial on Y_F automorphy factor $\pi^{-1}\varphi$

$$d \in \mathbb{Z}, \quad H^0(X_F, \mathcal{O}(d)) = \begin{cases} 0 & d < 0 \\ E & d = 0 \\ \mathcal{O}(Y)^{\varphi = \pi^d} & \end{cases}$$

∞-dim. Banach

E-space $d > 0$

Def: $\mathcal{X}_F = \mathrm{Proj}\Big(\underbrace{\bigoplus_{d\geq 0} \mathcal{O}(Y_F)^{\varphi=\pi^d}}_{\text{graded algebra of Fontaine's periods}}\Big)$

E-Scheme.

not of finite type!

Th: (1) $\mathcal{X}_F$ is a $\overbrace{\text{Dedekind scheme}}^{\text{a Curve}}$

(2) $\exists$ morphism of ringed spaces

GAGA type morphism

$$X_F \longrightarrow \mathcal{X}_F \quad \text{inducing}$$

$$|X_F|^{cl} \xrightarrow{\sim} |\mathcal{X}_F| = \text{closed points}$$

s.t. if $\tilde{x} \longmapsto x$ $\quad k(\tilde{x}) = k(x) = \text{perfectoid field} / E$

$$\widehat{\mathcal{O}}_{\mathcal{X}_F, x} \xrightarrow{\sim} \widehat{\mathcal{O}}_{X_F, \tilde{x}} = \underbrace{B^+_{dR}(k(x))}_{\text{Fontaine's period ring}} \quad \text{D.V.R.}$$

(3) $\forall f \in E(\mathcal{X}_F)^\times \quad \deg(\mathrm{div}\, f) = 0$

$x \in X_F, \quad \deg(x) := [k(x)^b : F]$

"The Curve is Complete"

$\Downarrow$

Good notion of degree of a line bundle + Harder–Narasimhan filtration for v.b./X_F.

(4) $\boxed{|\mathcal{X}_F|^{\deg=1} \xrightarrow{\sim} \{\text{untilts of } F \text{ over } E\}/\mathrm{Frob}.}$

$x \longmapsto k(x) + \text{iso. } F \xrightarrow{\sim} k(x)^b$

(5) If F alg. closed then $\forall x, \deg(x) = 1$.

$H^0(\mathcal{O}(1)) \setminus \{0\} / E^\times \xrightarrow{\sim} |\mathcal{X}_F|$

$t \longmapsto V^+(t)$

Moreover if $\{\infty\} = V^+(t)$ ← Fontaine's t

$$\mathcal{X}_F \setminus \{\infty\} = \mathrm{Spec}\Big(\underbrace{\mathcal{O}(Y_F)[\tfrac{1}{t}]^{\varphi=\pi}}_{\text{P.I.D.}}\Big)$$

$\left(\mathcal{O}(Y_F)[\tfrac{1}{t}]^{\varphi=\pi}, -\mathrm{ord}_\infty\right)$ not euclidean

$H^1(\mathcal{O}(-1)) \neq 0$ Contrary to $\mathbb{P}^1$: $(k[T], \deg)$

Curve shares some similarities with $\mathbb{P}^1$ but very different here.

(6) GAGA: $\text{v.b.}/\mathcal{X}_F \xrightarrow{\sim} \text{v.b.}/X_F$

Picard group: ~~x~~ F alg. closed.

$$\deg : \underset{\langle \mathcal{O}(1)\rangle}{\mathrm{Pic}(X_F)} \xrightarrow{\sim} \mathbb{Z}$$

$\mathrm{Pic}(X_F) = \langle \mathcal{O}(1) \rangle$

* Any F: $\left[\mathrm{Pic}^0(X_F) \xrightarrow{\sim} \mathrm{Hom}\left(\mathrm{Gal}(\overline{F}|F), E^\times\right) \right]$

$$[\mathcal{L}] \longmapsto \chi_{\mathcal{L}}$$

Sen-type descent result from $\widehat{\overline{F}}$ to F

where if $\beta : X_{\widehat{\overline{F}}} \to X_F$

$$u : \mathcal{O}_{X_{\widehat{\overline{F}}}} \xrightarrow{\sim} \beta^* \mathcal{L}$$

$$\chi_{\mathcal{L}}(\sigma) = u^{-\sigma} \circ u \in \Gamma(X_{\widehat{\overline{F}}}, \mathcal{O})^\times = E^\times$$

Rem: More general Narasimhan-Seshadri type statement

$$\{\text{slope } 0 \text{ semi-stable v. b.} / X_F\} \xrightarrow{\sim} \mathrm{Rep}_E\left(\mathrm{Gal}(\overline{F}|F)\right)$$

unsplit pro-étale cover $Y \to Y/\varphi^{\mathbb{Z}}$

* F alg. closed.

$E_n|E$ unramified degree n

$$Y_F/\varphi^{n\mathbb{Z}} = X_{F, E_n} = X_{F,E} \otimes_E E_n$$

$\mathbb{Z}/n\mathbb{Z}$ cover

$$\downarrow \pi_n$$

$$Y_F/\varphi^{\mathbb{Z}} = X_{F,E}$$

$\lambda = \frac{d}{h}$, $(d,h)=1$. $\underbrace{\mathcal{O}(\lambda)}_{\text{stable slope } \lambda} := \pi_{h*}\mathcal{O}_{X_{F,E_h}}(d)$

Th: (1) Any slope λ semi-stable v.b. $/X$ is isomorphic to $\mathcal{O}(\lambda)^m$ for some m

(2) The H.N. filtration of a v.b. $/X$ is split

(3) $\{\lambda_1 \geq \dots \geq \lambda_n \,/\, n \in \mathbb{N}, \lambda_i \in \mathbb{Q}\} \xrightarrow{\sim} \mathrm{Bun}_X/\sim$

$$(\lambda_1, \dots, \lambda_n) \longmapsto \left[\bigoplus_{i=1}^{n} \mathcal{O}(\lambda_i)\right]$$

Plenty of other results:

* Classification of G-bundles $/X$, G reductive group $/E$ $\rightsquigarrow$ link with Kottwitz set $B(G)$
* F alg. closed $\Rightarrow$ $X_{F,E}$ is géo. simply connected i.e. $\mathrm{Gal}(\bar{E}|E) \xrightarrow{\sim} \pi_1(X_{F,E})$

Moreover if Λ = finite abelian group

+ discrete action of $\mathrm{Gal}(\overline{E}/E)$

$\mathcal{F}$ = associated local system on X_F

$$R\Gamma_{\acute{e}t}(E, M) \xrightarrow{\sim} R\Gamma_{\acute{e}t}(X_F, \mathcal{F})$$

$$H^2(E, \mu_n) \xrightarrow{\sim} H^2(X_F, \mu_n) \xrightarrow{\sim} \mathbb{Z}/n\mathbb{Z}$$

$$\cup$$

fundamental class of local class field theory $\longleftrightarrow$ $\eta_X = c_1(\mathcal{O}(1))$ = fundamental class of the curve